Semigroups in finite von Neumann algebras

Gilles Cassier

Dedicated to the memory of Professor Béla Szőkefalvi-Nagy

Abstract. Let M be a finite von Neumann algebra. In the first part, we give asymptotic results about M-stable sequences of weak*-continuous mappings which are related with operators belonging to M. In the second part, we extend, by a shorter way, similarity results given in [CaFa2] to unbounded semigroups of operators contained in a finite von Neumann algebra.

I. Introduction and preliminaries

Let H be a separable complex Hilbert space and let $B(H)$ be the algebra of bounded linear operators acting on H. The ultra-weak topology of $B(H)$ is the weak* topology (in the sequel we will shorten weak* to w*) that comes from the well known duality $B(H) = (C_1(H))^*$, where $C_1(H)$ is the Banach space of trace class operators on H endowed with the trace norm (see [Dix]). A von Neumann algebra acting on H is by definition an ultra-weakly closed *-subalgebra of $B(H)$. Such a von Neumann algebra M is finite if it admits a faithful normal trace τ, which means that τ is an ultra-weakly continuous linear functional on M satisfying:

1) $\tau(AB) = \tau(BA)$ for any $A, B \in M$;

2) for any positive element A in M, we have $\tau(A) \geq 0$ and $\tau(A) = 0 \Longrightarrow A = 0$.

We denote by $\mathcal{T}(M)$ the set of all faithful normal traces acting on M. A good example of a finite von Neumann algebra is the w*-algebra generated by the left regular representation of a countable discrete group. We will denote by M_* the predual of M. For any subset $\mathcal{F}$ of M, we shall denote by $\mathcal{F}'$ the family of operators commuting with every element of $\mathcal{F}$.

Let $B(M)$ denote the algebra of bounded linear operators acting on M, and let $B_w(M)$ stand for the algebra of operators $T \in B(M)$ wich are weak*-continuous. Recall that $\phi \in B_w(M)$ if and only if ϕ is the adjoint of a bounded linear operator acting on the Banach space M_* (see for instance [BCP]). For any $\phi \in B_w(M)$, let ϕ_* denote the uniquely

determined operator whose (Banach space) adjoint is ϕ, that is $(\phi_*)^* = \phi$. For more details on von Neumann algebras, we refer the reader to [Dix] and [Sak].

As usual $[A_{i,j}]_{1\le i,j\le n} \in \mathcal{M}_n(B(H))$ denotes the $n\times n$ matrix which acts on the orthogonal sum of n copies of H; its entries are operators acting on H. We remind the reader that $\mathcal{M}_n(M)$ inherits a unique structure of von Neumann algebra. Let ψ be a linear mapping from M into itself, we define $\psi_n : \mathcal{M}_n(M) \to \mathcal{M}_n(M)$ by $\psi_n([A_{i,j}]_{1\le i,j\le n}) = [\psi(A_{i,j})]_{1\le i,j\le n}$. We call ψ n-positive if ψ_n is positive (that is positive operators are transformed into positive ones) and we call ψ completely positive if ψ is n-positive for all n.

We proved in [CaFa2] that a power bounded operator T in a finite von Neumann algebra M is similar to a unitary element of M if and only if $T^n x \not\to 0$ for any $x \in H\backslash\{0\}$ (T is said to be of class $C_{1\cdot}$ in the Sz.-Nagy–Foias terminology). We will extend this result into two directions. On the one hand, we will consider general semigroups. On the other hand, we will work with operators which are not necessarily power bounded. To achieve this, we have to find a proper framework, which will allow short and well adapted methods.

In similarity problems, the idea of using limits in the sense of Banach comes from B. Sz.-Nagy [Nag]. In the sequel, we frequently use this idea. Recall that a Banach limit is a state, that is a linear functional L with $\|L\| = L(\mathbf{1}) = 1$, acting on the classical space ℓ^∞ of all complex bounded sequences and satisfying $L((u_{n+1})) = L((u_n))$. A bounded sequence $(u_n)_{n\ge1}$ is said to be almost convergent to a complex number c if

$$\lim_{n\to\infty} \sup_{k\in\mathbf{N}} \left| \frac{1}{n+1} \sum_{i=k}^{k+n} u_i - c \right| = 0.$$

Lorentz proved in [Lor] that $(u_n)_{n\ge1}$ is almost convergent to c if and only if for every Banach limit L we have $c = L((u_n))$. A sequence $(u_n)_{n\ge1}$ is said to be strongly almost convergent to c if the sequence $(|u_n - c|)_{n\ge1}$ is almost convergent to 0. We will say that a sequence $(\phi_n)_{n\ge1}$ of operators in $B_w(M)$ is weakly almost convergent to $\phi \in B_w(M)$ if $[l, \phi_n(X) - \phi(X)]$ almost converges to 0 for any $(l, X) \in M_* \times M$.

Definition 1.1. A mapping $p\colon \mathbf{N} \to (0,\infty)$ is called a *gauge* if there exists $c_p > 0$ such that the sequence $p(n+1)/p(n)$ is strongly almost convergent to c_p. Moreover, if in addition the sequence $c_p^n/p(n)$ strongly almost converges to 1, then we say that p is a *regular gauge*.

We will say that a sequence $(T_n)_{n\ge1}$ of operators, acting on a Banach space, is *dominated* by a gauge p if $\|T_n\| \le p(n)$ holds for every positive integer n. We follow [Ker] in

saying that $(T_n)_{n\geq 1}$ is *compatible* with a gauge p if in addition the sequence $\|T_n\|/p(n)$ does not almost converge to 0. An operator T is dominated by (compatible with) p if the sequence $(T^n)_{n\geq 1}$ is dominated by (resp. compatible with) p. Finally, a family $\mathcal{F}$ of operators is called dominated by (compatible with) p if each operator in $\mathcal{F}$ is dominated by (resp. compatible with) p. For some recent contributions in this area, we refer the reader to [Ker], [Ker1], [Ker2], [Ker3], [Ker4], [Ker5] and [KeMü].

Assume that p is a gauge and $T \in M$ is dominated by p. Given a Banach limit L, let us introduce the (bounded, linear) operator $E_{L,T}$, acting on M, by setting

$$[l, E_{L,T}(X)] = L(\{[l, T^{*n}XT^n]p(n)^{-2}\}_{n\geq 1})$$

for any $(l, X) \in M_* \times M$. The following proposition summarizes some useful properties of the operator $E_{L,T}$.

Proposition 1.2. *Let T be an element in a von Neumann algebra M acting on a separable Hilbert space H. Assume that T is dominated by a gauge p. Then, for any Banach limit L, we have*

(i) *$E_{L,T}$ is a completely positive mapping*;

(ii) *$E_{L,T}(T^*XT) = c_p^2 E_{L,T}(X)$ for any $X \in M$*;

(iii) *if $A, B \in M$ commute with T, then we have $E_{L,T}(A^*XB) = A^*E_{L,T}(X)B$ for any $X \in M$*;

(iv) *$T^*E_{L,T}(X)T = c_p^2 E_{L,T}(X)$ for any $X \in M$*;

(v) *there exists $\rho_L(p) \in [0,1]$ such that $E_{L,T} \circ E_{L,T} = \rho_L(p)E_{L,T}$*;

(vi) *moreover, if M is a finite von Neumann algebra, then the mapping $E_{L,T}$ belongs to $B_w(M)$.*

Remark 1.3. If $T \in M$ is compatible with a gauge p, then the spectral radius $r(T)$ satisfies $r(T) = c_p$ (see [Ker)]).

Proof. (i) Let $[X_{i,j}]_{1\leq i,j\leq p}$ be a positive $p \times p$ matrix whose entries are operators in M and let $x_1, ..., x_n$ be vectors in H. For any $(i,j) \in \{1,..,p\}^2$, we define the linear functional $l_{i,j}$ acting on M by setting $l_{i,j}(X) = \langle Xx_j, x_i\rangle$. It is obvious that $l_{i,j} \in M_*$, hence

$$\left\langle \begin{bmatrix} E_{L,T}(X_{1,1}) & \cdots & E_{L,T}(X_{1,p}) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ E_{L,T}(X_{p,1}) & \cdots & E_{L,T}(X_{p,p}) \end{bmatrix} \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_p \end{bmatrix}, \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_p \end{bmatrix} \right\rangle = \sum_{i,j=1}^{p} \langle E_{L,T}(X_{i,j})x_j, x_i\rangle$$

$$= L\left(\sum_{i,j=1}^{p}(\{[l_{i,j}, T^{*n}X_{i,j}T^n]p(n)^{-2}\}_{n\geq 1}\right)$$

$$= L\left(\left\{\left\langle\begin{bmatrix} X_{1,1} & \cdots & X_{1,p} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ X_{p,1} & \cdots & X_{p,p}\end{bmatrix}\begin{bmatrix} T^n x_1 \\ \cdot \\ \cdot \\ \cdot \\ T^n x_p\end{bmatrix}, \begin{bmatrix} T^n x_1 \\ \cdot \\ \cdot \\ \cdot \\ T^n x_p\end{bmatrix}\right\rangle p(n)^{-2}\right\}_{n\geq 1}\right) \geq 0.$$

The positivity of the last term follows from the positivity of the matrix $[X_{i,j}]_{1\leq i,j\leq p}$ and the positivity of the state L.

(ii) Given any l in M_*, we have

$$[l, E_{L,T}(T^*XT)] = L(\{[l, T^{*n+1}XT^{n+1}]p(n)^{-2}\}_{n\geq 1})$$

$$= L\left(\left\{\frac{[l, T^{*n+1}XT^{n+1}]}{p(n+1)^2}\frac{p(n+1)^2}{p(n)^2}\right\}_{n\geq 1}\right).$$

Since p is a gauge, we see that the sequence $(|p(n+1)/p(n) - c_p|)_{n\geq 1}$ is almost convergent to 0. It follows that $(\left|p(n+1)^2/p(n)^2 - c_p^2\right|)_{n\geq 1}$ also almost converges to 0. By Lemma 1 from [Ker], we get

$$[l, E_{L,T}(T^*XT)] = c_p^2 L(\{[l, T^{*n+1}XT^{n+1}]p(n+1)^{-2}\}_{n\geq 1})$$

$$= c_p^2 L(\{[l, T^{*n}XT^n]p(n)^{-2}\}_{n\geq 1}) = c_p^2[l, E_{L,T}(X)]$$

and (ii) follows.

(iii) Let A, B be two operators in M commuting with T, we have

$$[l, E_{L,T}(A^*XB)] = L(\{[l, T^{*n}A^*XBT^n]p(n)^{-2}\}_{n\geq 1})$$

$$= L(\{[l, A^*T^{*n}XT^nB]p(n)^{-2}\}_{n\geq 1}) = [l, A^*E_{L,T}(X)B].$$

This establishes the formula.

(iv) follows immediately from (ii) and (iii).

(v) Let $l \in M_*$; using (iv), we get

$$[l, E_{L,T}(E_{L,T}(X))] = L(\{[l, T^{*n}E_{L,T}(X)T^n]p(n)^{-2}\}_{n\geq 1})$$

$$= L\left(\left\{c_p^{2n}p(n)^{-2}\right\}_{n\geq 1}\right)[l, E_{L,T}(X)] = \rho_L(p)[l, E_{L,T}(X)],$$

by setting $\rho_L(p) = L(\{c_p^{2n} p(n)^{-2}\}_{n\geq 1})$. From the formula $c_p = \inf\{p(n)^{1/n} : n \in \mathbf{N}\}$ (see [Ker] Proposition 1), we immediately deduce that $\rho_L(p) \in [0,1]$.

(vi) It suffices to show that the linear functional $M \ni X \mapsto l(E_{L,T}(X))$ is ultra-weakly continuous for any $l \in M_*$. Let Z be in M, for clarity we will denote by l_Z the element in M_* given by $l_Z(X) = \tau(ZX)$ for any $X \in M$. Given $X, Y \in M$, we have

$$\tau(E_{L,T}(X)Y) = [l_Y, E_{L,T}(X)] = L(\{[l_Y, T^{*n}XT^n]p(n)^{-2}\}_{n\geq 1})$$

$$= L(\{\tau(YT^{*n}XT^n)p(n)^{-2}\}_{n\geq 1}) = L(\{\tau(XT^nYT^{*n})p(n)^{-2}\}_{n\geq 1})$$

$$= [l_X, E_{L,T^*}(Y)] = \tau(XE_{L,T^*}(Y)),$$

hence

$$\tau(E_{L,T}(X)Y) = \tau(XE_{L,T^*}(Y)).$$

Let Y be in M, we deduce from the last equation that $l_Y \circ E_{L,T}$ is ultra-weakly continuous. Since the linear functionals l_Y with $Y \in M$ are dense in M_*, it follows that $E_{L,T}$ is ultra-weakly continuous. This completes the proof. Q.E.D.

II. Convergence of τ-M-stable maps

Given any $X \in M$, the linear functional $l_X(Y) := \tau(XY)$ $(Y \in M)$ is weak*-continuous, and so $l_X \in M_*$. The mapping

$$\Psi_\tau \colon M \to M_*, \; X \mapsto l_X$$

is a bounded linear quasiaffinity; the linear manifold $\widehat{M}_\tau := \mathrm{ran}\Psi$ is dense in M_*.

Let us consider the set

$$B_\tau(M) := \{\phi \in B_w(M) : \phi_*(\widehat{M}_\tau) \subset \widehat{M}_\tau\}$$

of τ-M-stable weak*-continuous operators. For any $\phi \in \widehat{B}_\tau(M)$, we can introduce the linear mapping

$$\widehat{\phi}_\tau := \Psi_\tau^{-1}\phi_*\Psi_\tau \colon M \to M.$$

For any $X, Y \in M$, we have

$$\tau(X\phi(Y)) = [l_X, \phi(Y)] = [\phi_*(l_X), Y] = [l_{\widehat{\phi}_\tau(X)}, Y] = \tau(\widehat{\phi}_\tau(X)Y).$$

An application of the Closed Graph Theorem yields that $\widehat{\phi}_\tau$ is bounded. In fact, we see that $\widehat{\phi}_\tau$ is also in $\widehat{B}_\tau(M)$ and we have $\widehat{\left(\widehat{\phi}_\tau\right)}_\tau = \phi$. We say that ϕ is M-stable if it is τ-M-stable for every $\tau \in \mathcal{T}(M)$. We will consider the set

$$\widehat{B}(M) = \bigcap_{\tau \in \mathcal{T}(M)} \widehat{B}_\tau(M)$$

of all M-stable operators.

Remarks 2.1. **1.** Denote by $\mathcal{M}_p(\mathbf{C})$ the algebra of square matrices of order p, and consider the finite von Neumann algebra $M = \oplus_{p\geq 2}\mathcal{M}_p(\mathbf{C})$ acting on the Hilbert space $H = \oplus_{p\geq 2}\mathbf{C}^p$ in an obvious sense. We consider the faithful normal traces τ_1 and τ_2 defined by setting

$$\tau_1(\oplus_{p\geq 2}X_p) = \sum_{p\geq 2}\frac{1}{p^3}\mathrm{Tr}(X_p),$$

$$\tau_2(\oplus_{p\geq 2}X_p) = \sum_{p\geq 2}\alpha_p \mathrm{Tr}(X_p),$$

where $\mathrm{Tr}(\cdot)$ is the usual trace acting on $\mathcal{M}_p(\mathbf{C})$ and α_p is given by

$$\alpha_p = \begin{cases} \frac{1}{2^p} & \text{if} \quad p \notin 3^{\mathbf{N}} \\ \frac{p}{2^p} & \text{if} \quad p \in 3^{\mathbf{N}}. \end{cases}$$

Let us consider the mapping ϕ defined by

$$\phi(X_2, X_3, ...) = (0, X_2', X_3', ...)$$

where $X_p' \in \mathcal{M}_{p+1}(\mathbf{C})$ is given in an obvious sense by

$$X_p' = \begin{bmatrix} X_p & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, we can check that $\phi \in \widehat{B}_{\tau_1}(M)$ but $\phi \notin \widehat{B}_{\tau_2}(M)$.

2. If M is a factor ($M \cap M' = \mathbf{C}I$), then all faithful normal traces are proportional (see [Dix, p. 249]). Consequently we have $\widehat{B}(M) = \widehat{B}_\tau(M)$ for every $\tau \in \mathcal{T}(M)$.

3. Let M be a finite von Neumann algebra and $A, B \in M$, then the mapping $\phi\colon X \mapsto AXB$ belongs to $\widehat{B}(M)$.

Let M be a finite von Neumann algebra. Recall that $\mathcal{M}_n(M)$ is also a finite von Neumann algebra with the faithful normal trace τ_n defined by setting

$$\tau_n([A_{i,j}]_{1\leq i,j\leq n}) = \sum_{k=1}^{n} \tau(A_{k,k}).$$

We begin with some useful properties of the operators $\widehat{\phi}$ when ϕ is a τ-M-stable mapping.

Proposition 2.2. *Let M be a finite von Neumann algebra, τ a faithful normal trace on M and $\phi \in B(M)$. Then*

(i) *the mapping ϕ belongs to $\widehat{B}_\tau(M)$ if and only if there exists $\psi \in B(M)$ such that $\tau(\phi(X)Y) = \tau(X\psi(Y))$ for every $X, Y \in M$; and then $\psi = \widehat{\phi}_\tau$;*

(ii) *the set $\widehat{B}_\tau(M)$ is an algebra; the mapping $\phi \mapsto \widehat{\phi}_\tau$ is linear, involutive and $(\widehat{\phi_1\phi_2})_\tau = (\widehat{\phi}_1)_\tau(\widehat{\phi}_2)_\tau$;*

(iii) *if $\phi \in \widehat{B}_\tau(M)$, then $\phi_n \in \widehat{B}_\tau(\mathcal{M}_n(M))$ and we have $(\widehat{\phi_n})_\tau = (\widehat{\phi}_\tau)_n$;*

(iv) *if ϕ is n-positive ($n \in \mathbf{N}$), then $\widehat{\phi}_\tau$ is also n-positive*;

(v) *if ϕ is completely positive, then $\widehat{\phi}_\tau$ is completely positive*;

(vi) *assume that ϕ is 2-positive, then the mappings ϕ and $\widehat{\phi}_\tau$ extend uniquely to bounded operators from $L^2(M,\tau)$ into itself; moreover, we have*

$$\|\phi\|_{B(L^2(M,\tau))} = \left\|\widehat{\phi}_\tau\right\|_{B(L^2(M,\tau))} \leq \left(\|\phi(I)\|_M\right)^{1/2}\left(\left\|\widehat{\phi}_\tau(I)\right\|_M\right)^{1/2}.$$

Proof. (i) If $\phi \in \widehat{B}_\tau(M)$, it suffices to set $\psi = \widehat{\phi}_\tau$. Conversely, assume that there exists $\psi \in B(M)$ such that $\tau(\phi(X)Y) = \tau(X\psi(Y))$ for every $X, Y \in M$. We immediately deduce that the linear functional $X \mapsto \tau(\phi(X)Y)$ is ultra-weakly continuous for each $Y \in M$. Since $\widehat{M}_\tau$ is dense in M_*, we see that $\phi \in B_w(M)$. Moreover, we have

$$\phi_*(l_X) = l_{\psi(X)}$$

for any $X \in M$, thus we have $\phi_*(\widehat{M}_\tau) \subset \widehat{M}_\tau$. This gives $\phi \in \widehat{B}_\tau(M)$.

(ii) This statement follows clearly from the characterization of elements of $\widehat{B}_\tau(M)$ given in (i).

(iii) Assume that $\phi \in \widehat{B}_\tau(M)$. Let $A = [A_{i,j}]_{1\leq i,j\leq n}$ and $B = [B_{i,j}]_{1\leq i,j\leq n}$ be two elements in $\mathcal{M}_n(M)$, then

$$\tau_n((\widehat{\phi})_n(A)B) = \tau_n([\widehat{\phi}(A_{i,j})]_{1\leq i,j\leq n}[B_{i,j}]_{1\leq i,j\leq n}) = \sum_{k=1}^n \tau\left(\sum_{l=1}^n \widehat{\phi}(A_{k,l})B_{l,k}\right)$$

$$= \sum_{k=1}^n\sum_{l=1}^n \tau(\widehat{\phi}(A_{k,l})B_{l,k}) = \sum_{k=1}^n\sum_{l=1}^n \tau(A_{k,l}\phi(B_{l,k}))$$

$$= \tau_n(A\phi_n(B)).$$

It follows easily by (i) that $\phi_n \in \widehat{B}_\tau(\mathcal{M}_n(M))$ and we have $\widehat{\phi_n} = (\widehat{\phi})_n$.

(iv) Assume that $\phi \in \widehat{B}_\tau(M)$ is positive. Let A and B be two positive elements in M, we have

$$\tau(\widehat{\phi}_\tau(A)B) = \tau(A\phi(B)) = \tau(\sqrt{A}\phi(B)\sqrt{A}) \geq 0.$$

Hence, we derive easily the positivity of $\widehat{\phi}_\tau$ from the previous calculation. If ϕ is n-positive, the map ϕ_n is positive, thus $(\widehat{\phi_n})_\tau$ is positive and the formula $(\widehat{\phi_n})_\tau = (\widehat{\phi}_\tau)_n$ implies that $\widehat{\phi}_\tau$ is n-positive.

(v) It is clear from (iv) that $\widehat{\phi}$ is completely positive if ϕ is completely positive.

(vi) Let $\phi \in \widehat{B}_\tau(M)$ be 2-positive and $Y \in M$, then the matrix

$$\begin{bmatrix} \phi(I) & \phi(Y) \\ \phi(Y)^* & \phi(Y^*Y) \end{bmatrix}$$

is positive, a fact which implies that

$$\phi(Y)^*\phi(Y) \leq \|\phi(I)\|\,\phi(Y^*Y).$$

Given a pair (X, Y) of elements of M, we deduce from the previous inequality that

$$\left|\tau(\widehat{\phi}_\tau(X)Y)\right| = |\tau(X\phi(Y))| \leq \sqrt{\tau(X^*X)}\sqrt{\tau(\phi(Y)^*\phi(Y))}$$

$$\leq \sqrt{\|\phi(I)\|}\sqrt{\tau(X^*X)}\sqrt{\tau(\phi(Y^*Y))}$$

$$= \sqrt{\|\phi(I)\|}\|X\|_2\sqrt{\tau(Y^*Y\widehat{\phi}_\tau(I))}$$

$$\leq \sqrt{\|\phi(I)\|}\sqrt{\left\|\widehat{\phi}_\tau(I)\right\|}\|X\|_2\|Y\|_2.$$

It follows that $\|\widehat{\phi}_\tau(X)\|_2 \leq \sqrt{\|\phi(I)\|}\sqrt{\left\|\widehat{\phi}_\tau(I)\right\|}\|X\|_2$. Using the density of M in $L^2(M, \tau)$, we see that the map $\widehat{\phi}_\tau$ extends uniquely to a bounded operator from $L^2(M, \tau)$ into itself. We also get

$$\left\|\widehat{\phi}\right\|_{B(L^2(M))} \leq \sqrt{\|\phi(I)\|_M} \cdot \sqrt{\left\|\widehat{\phi}_\tau(I)\right\|_M}.$$

Observe that the adjoint of ϕ in $L^2(M, \tau)$ is given by $\phi^*(X) = \widehat{\phi}_\tau(X^*)^*$ for any $X \in M$. The rest of the proof follows immediately. Q.E.D.

Remark 2.3. It follows immediately from (ii) that $\widehat{B}(M)$ is also an algebra.

Let $\Phi = (\phi_n)_{n\geq 1}$ be a sequence in $B(M)$ dominated by a gauge p. Given a Banach limit L, let us consider the limit operator $E_{\Phi,L} \in B(M)$, defined by

$$[l, E_{\Phi,L}(X)] = L(\{[l, \phi_n(X)]p(n)^{-1}\}_{n\geq 1})$$

for any $(l, X) \in M_* \times M$. Note that the previous formulas actually define $E_{\Phi,L}$ as an element of $B(M)$. We write $\gamma_L(p) = L(\{c_p^n p(n)^{-1}\}_n)$.

The following theorem seems to be of independent interest. It presents some fine properties of abelian sequences included in $\widehat{B}_\tau(M)$ which are compatible with a gauge p, where M is a finite von Neumann algebra and τ is a faithful normal trace on M.

Theorem 2.4. *Let M be a finite von Neumann algebra, τ a faithful normal trace on M and $\Phi = (\phi_n)_{n\geq 1}$ a sequence in $\widehat{B}_\tau(M)$ dominated by a gauge p and such that $\widehat{\Phi}_\tau = ((\widehat{\phi_n})_\tau)_{n\geq 1}$ is also dominated by p.*

(i) *The operator $E_{\Phi,L}$ belongs to $\widehat{B}_\tau(M)$ and we have $(\widehat{E_{\Phi,L}})_\tau = E_{\widehat{\Phi}_\tau,L}$ for any Banach limit L.*

(ii) *Suppose $\Phi = (\phi_n)_{n\geq 1}$ is abelian, then the operators E_{Φ,L_1} and E_{Φ,L_2} commute for any pair (L_1, L_2) of Banach limits.*

(iii) *Assume that $\psi \in \widehat{B}_\tau(M)$ and that the sequences $\phi_n = \psi^n$ and $\widehat{\phi_n} = \widehat{\psi}^n$ are dominated by the gauge p. Then we have*

$$E_{\Phi,L_2} \circ E_{\Phi,L_1} = E_{\Phi,L_1} \circ E_{\Phi,L_2} = \gamma_{L_1}(p)E_{\Phi,L_2} = \gamma_{L_2}(p)E_{\Phi,L_1}$$

for any Banach limits L_1 and L_2. In particular, if $c_p^n/p(n)$ almost converges to a nonzero limit, then $(p(n)^{-1}\phi_n)_{n\geq 1}$ weakly almost converges to an operator ϕ belonging to $\widehat{B}_\tau(M)$.

Proof. (i) Given a pair (X, Y) of elements of M, we get

$$\tau(E_{\Phi,L}(X)Y) = [l_Y, E_{\Phi,L}(X)] = L(\{[l_Y, \phi_n(X)]p(n)^{-1}\}_{n\geq 1})$$

$$= L(\{\tau(\phi_n(X)Y)p(n)^{-1}\}_{n\geq 1}) = L(\{\tau(X(\widehat{\phi_n})_\tau(Y))p(n)^{-1}\}_{n\geq 1})$$

$$= L(\{[l_X, (\widehat{\phi_n})_\tau(Y)]p(n)^{-1}\}_{n\geq 1}) = [l_X, E_{\hat{\Phi}_\tau,L}(Y)] = \tau(XE_{\hat{\Phi}_\tau,L}(Y))$$

for every Banach limit L. Now Proposition 2.2.(i) implies the statement.

(ii) Assume that the sequence $(\phi_n)_{n\geq 1}$ is abelian. We have

$$[l_{(\widehat{\phi_m})_\tau(Y)}, \phi_n(X)] = \tau(\phi_n(X)(\widehat{\phi_m})_\tau(Y)) = \tau(\phi_m(X)(\widehat{\phi_n})_\tau(Y)) = [l_{\phi_m(X)}, (\widehat{\phi_n})_\tau(Y)]$$

for any pair (m,n) of positive integers and any pair $(X,Y)\in M^2$. By taking L_1-limit with respect to n we get

$$[l_{E_{\Phi,L_1}(X)},(\widehat{\phi_m})_\tau(Y)]=\tau(E_{\Phi,L_1}(X)(\widehat{\phi_m})_\tau(Y))=[l_{(\widehat{\phi_m})_\tau(Y)},E_{\Phi,L_1}(X)]$$

$$=[l_{\phi_m(X)},E_{\hat{\Phi}_\tau,L_1}(Y)]=\tau(\phi_m(X)E_{\hat{\Phi}_\tau,L_1}(Y))=[l_{E_{\hat{\Phi}_\tau,L_1}(Y)},\phi_m(X)].$$

Now, taking L_2-limit with respect to m and using (i), we obtain that

$$\tau(E_{\Phi,L_2}\circ E_{\Phi,L_1}(X)Y)=\tau(E_{\Phi,L_1}(X)(\widehat{E_{\Phi,L_2}})_\tau(Y))=\tau(E_{\Phi,L_1}(X)E_{\hat{\Phi}_\tau,L_2}(Y))$$

$$=[l_{E_{\Phi,L_1}(X)},E_{\hat{\Phi}_\tau,L_2}(Y)]=[l_{E_{\hat{\Phi},L_1}(Y)},E_{\Phi,L_2}(X)]=\tau(E_{\hat{\Phi}_\tau,L_1}(Y)E_{\Phi,L_2}(X))$$

$$=\tau((\widehat{E_{\Phi,L_1}})_\tau(Y)E_{\Phi,L_2}(X))=\tau(E_{\Phi,L_1}\circ E_{\Phi,L_2}(X)Y).$$

We thus have $E_{\Phi,L_2}\circ E_{\Phi,L_1}=E_{\Phi,L_1}\circ E_{\Phi,L_2}$.

(iii) Let $\psi\in B_w(M)$ and (L_1,L_2) be a pair of Banach limits. For any $X,Y\in M$, we have

$$[l_{\phi_n(X)},(\widehat{\phi_m})_\tau(Y)]p(n)^{-1}p(m)^{-1}=\tau(\phi_n(X)(\widehat{\phi_m})_\tau(Y))p(n)^{-1}p(m)^{-1}$$

$$=\tau(\psi^{m+n}(X)Y)p(n)^{-1}p(m)^{-1}=\frac{p(m+n)}{p(m)p(n)}[l_Y,\psi^{m+n}(X)]p(m+n)^{-1}.$$

The sequence $p(m+n)/p(m)=p(m+1)/p(m)...p(m+n)/p(m+n-1)$ is strongly almost convergent to c_p^n, when m goes to infinity. By taking L_1-limit with respect to m, we get

$$[l_{E_{\hat{\Phi}_\tau,L_1}(Y)},\phi_n(X)]p(n)^{-1}=\tau(\phi_n(X)E_{\hat{\Phi}_\tau,L_1}(Y))p(n)^{-1}$$

$$=[l_{\phi_n(X)},E_{\hat{\Phi}_\tau,L_1}(Y)]p(n)^{-1}=c_p^n p(n)^{-1}[l_Y,E_{\Phi,L_1}(X)].$$

After taking L_2-limit with respect to n, it follows that

$$\tau(E_{\Phi,L_1}\circ E_{\Phi,L_2}(X)Y)=[l_{E_{\hat{\Phi}_\tau,L_1}(Y)},E_{\Phi,L_2}(X)]$$

$$=\gamma_{L_2}(p)[l_Y,E_{\Phi,L_1}(X)]=\gamma_{L_2}(p)\tau(E_{\Phi,L_1}(X)Y),$$

whence $E_{\phi,L_1}\circ E_{\phi,L_2}=\gamma_{L_2}(p)E_{\phi,L_1}$. We deduce from (ii) that E_{ϕ,L_1} and E_{ϕ,L_2} commute. Interchanging the role of L_1 and L_2 we conclude that

$$E_{\Phi,L_2}\circ E_{\Phi,L_1}=E_{\Phi,L_1}\circ E_{\Phi,L_2}=\gamma_{L_1}(p)E_{\Phi,L_2}=\gamma_{L_2}(p)E_{\Phi,L_1}.$$

When the sequence $c_p^n/p(n)$ almost converges to a nonzero number, then we deduce that the limit of $(p(n)^{-1}\phi_n)_{n\geq 1}$ is independent of L. Applying Lorentz's result, we obtain that $(p(n)^{-1}\phi_n)_{n\geq 1}$ is weakly almost convergent to an operator in $\widehat{B}_\tau(M)$. The proof is now complete. Q.E.D.

Remarks 2.5. 1. If $c_p^n p(n)^{-1}$ is almost convergent to a nonzero limit, then we see that the Cesaro means $(n+1)^{-1}(p(0)^{-1}\phi_0+...+p(n)^{-1}\phi_n)$ weakly converge to an operator in $B_w(M)$ (which is obviously ultra-weakly continuous).

2. The assumption that $\Phi = (\phi_n)_{n\geq 1}$ is a sequence in $\widehat{B}_\tau(M)$ dominated by a gauge p does not imply that $\widehat{\Phi}_\tau = ((\widehat{\phi_n})_\tau)_{n\geq 1}$ is also dominated by p. Denote by $\mathcal{M}_p(\mathbf{C})$ the algebra of square matrices of order p, and consider the finite von Neumann algebra $M = \oplus_{p\geq 2}\mathcal{M}_p(\mathbf{C})$ acting on the Hilbert space $H = \oplus_{p\geq 2}\mathbf{C}^p$ in an obvious sense. We consider the faithful normal trace τ defined by

$$\tau(\oplus_{p\geq 2}X_p) = \sum_{p\geq 2}\frac{1}{p^3}\mathrm{Tr}(X_p),$$

where $\mathrm{Tr}(\cdot)$ is the usual trace on $\mathcal{M}_p(\mathbf{C})$. Fix a unit vector e_p in $\mathbf{C}^p$ and write P_p for the orthogonal projection onto $\mathbf{C}^p \ominus \mathbf{C}e_p$. For any $n \geq 2$, set $\phi_n(\oplus_{p\geq 2}X_p) = \langle X_n e_n, e_n\rangle P_n$. We can easily see that $\Phi = (\phi_n)_{n\geq 1}$ is a sequence in $\widehat{B}_\tau(M)$ such that $\|\phi_n\| = 1$ for every n. Hence Φ is dominated by the constant gauge p equal to 1, but $\widehat{\Phi}_\tau = ((\widehat{\phi_n})_\tau)_{n\geq 1}$ is not dominated by p , actually $\|(\widehat{\phi_n})_\tau\| = n-1$.

Let $\mathcal{F}$ be an abelian set included in $\widehat{B}_\tau(M)$. We consider the (abelian) semigroup $\mathcal{E}(\mathcal{F})$ induced by $\mathcal{F}$, that is

$$\mathcal{E}(\mathcal{F}) = \{\phi_1\circ\ldots\circ\phi_n : \phi_1,...,\phi_n \in \mathcal{F}, n\in\mathbf{N}\}.$$

We define a partial ordering on $\mathcal{E}(\mathcal{F})$ by setting $F \leq F'$ if there exists F'' in $\mathcal{E}(\mathcal{F})$ such that $F' = F''F$. (It is clear that $\mathcal{E}(\mathcal{F})$ is a directed set with this partial ordering, and it can be considered as a net (generalized sequence) indexed by itself.)

Proposition 2.6. *Let M be a finite von Neumann algebra, τ a faithful normal trace on M, and let $\mathcal{F}$ be an abelian set of m-positive projections belonging to $\widehat{B}_\tau(M)$ ($m\geq 2$). Assume that $\mathcal{E}(\mathcal{F})$ and $\widehat{\mathcal{E}(\mathcal{F})}$ are bounded in $B(M)$. Then the net $\mathcal{E}(\mathcal{F})$ weakly converges to an m-positive projection $E \in \widehat{B}_\tau(M)$.*

Proof. Let us introduce the classical Hilbert space $L^2(M,\tau)$ equipped with the inner product $\langle X, Y\rangle := \tau(Y^*X)$ $(X, Y \in M)$. Since every $\phi \in \mathcal{E}(\mathcal{F})$ is 2-positive, Proposition

2.2.(vi) shows that ϕ extends uniquely to a bounded projection (still denoted by ϕ) from $L^2(M,\tau)$ into itself. Using again Proposition 2.2.(vi), we see that the set $\mathcal{E}(\mathcal{F})$ is bounded in $B(L^2(M,\tau))$, thus it is weakly relatively compact.

Choose two cofinal subnets $(E_i)_{i\in\mathcal{I}}$ and $(E_j)_{j\in\mathcal{J}}$ in $\mathcal{E}(\mathcal{F})$, which converge respectively to F and G in the weak operator topology of $B(L^2(M,\tau))$. Fix $i\in\mathcal{I}$ and consider the set $\mathcal{J}_i=\{j\in\mathcal{J}: j\geq i\}$. Then we have

$$\langle E_i(X),E_j^*(Y)\rangle=\tau(E_i(X)(\widehat{E_j})_\tau(Y^*))=\tau(E_j\circ E_i(X)Y^*)$$

$$=\tau(E_j(X)Y^*)=\langle E_j(X),Y\rangle$$

for any $j\in\mathcal{J}_i$ and any pair $(X,Y)\in M^2$. Thus, taking limit with respect to the set $\mathcal{J}_i$, we obtain

$$\langle E_i(X),G^*(Y)\rangle=\langle G(X),Y\rangle.$$

Now, taking limit with respect to the directed set $\mathcal{I}$, we get

$$\langle G\circ F(X),Y\rangle=\langle F(X),G^*(Y)\rangle=\langle G(X),Y\rangle,$$

whence $G\circ F=G$ follows. Interchanging the role of F and G, we see that $F\circ G=F$. Since F and G are limit points of elements belonging to the commutative set $\mathcal{E}(\mathcal{F})$, they commute. Hence $F=F\circ G=G\circ F=G$, in particular $F=F\circ G=F\circ F$. We deduce that $\mathcal{E}(\mathcal{F})$ is weakly convergent in $B(L^2(M,\tau))$ to a projection E.

Now, we want to show that $E(M)\subset M$ and $E|M\in B(M)$. To this order let $X\in M$ be arbitrary, and let us consider the linear functional $\varphi(Y):=\langle E(X),Y^*\rangle$ $(Y\in M)$. Choosing a cofinal subnet $\{E_i\}_{i\in\mathcal{I}}$ in $\mathcal{E}(\mathcal{F})$, we have $\langle E(X),Y^*\rangle=\lim_i\langle E_i(X),Y^*\rangle=\lim_i\tau(E_i(X)Y)$. Thus

$$|\varphi(Y)|=\lim_i|\tau(E_i(X)Y)|\leq\liminf_i\|E_i(X)\|\|Y\|_1\leq C\|X\|\|Y\|_1,$$

where $C=\sup\{\|F\|: F\in\mathcal{E}(\mathcal{F})\}<\infty$ and $\|Y\|_1:=\tau(|Y|)=\|l_Y\|$ (see [Dix] Section I.6.10). We deduce that there exists unique $\tilde{X}\in M$ such that

$$\langle E(X),Y^*\rangle=\varphi(Y)=(\varphi\circ\Psi_\tau^{-1})(l_Y)=[l_Y,\tilde{X}]=\tau(\tilde{X}Y)=\langle\tilde{X},Y^*\rangle$$

holds for every $Y\in M$. Hence $E(X)=\tilde{X}\in M$ and $\|E(X)\|=\|\tilde{X}\|\leq C\|X\|$.

It is clear that

$$[l_Y,E(X)]=\tau(E(X)Y)=\langle E(X),Y^*\rangle=\lim_i\langle E_i(X),Y^*\rangle=\lim_i[l_Y,E_i(X)]$$

is true for every $X, Y \in M$. Since $\widehat{M}_\tau$ is dense in M_*, and $\mathcal{E}(\mathcal{F})$ is bounded, it follows that $\mathcal{E}(\mathcal{F})$ weakly converges to E.

We can prove in the same manner that $\widehat{\mathcal{E}(\mathcal{F})}$ converges weakly to an operator $F \in B(M)$. Taking into account that $\tau(E_i(X)Y) = \tau(X\widehat{(E_i)}_\tau(Y))$, we obtain by passing to the limit that

$$\tau(E(X)Y) = \tau(XF(Y))$$

holds for every $X, Y \in M$. It follows by Proposition 2.2.(i) that E belongs to $B_w(M)$.

It remains to prove that E is m-positive. It is clear that every operator in $\mathcal{E}(\mathcal{F})$ is m-positive. Let $[X_{k,l}]_m \in \mathcal{M}_m(M)$ be a positive operator. Given any vector $x = x_1 \oplus \cdots \oplus x_m \in H^{(m)}$, we have

$$\left\langle [E(X_{k,l})]_m \, x, x \right\rangle = \sum_{k,l=1}^{m} [l_{x_l,x_k}, E(X_{k,l})]$$

$$= \lim_i [l_{x_l,x_k}, E_i(X_{k,l})] = \lim_i \left\langle [E_i(X_{k,l})]_m \, x, x \right\rangle \geq 0,$$

and so E is an m-positive projection. Q.E.D.

Let M be a finite von Neumann algebra and let T be an operator in M dominated by the regular gauge p. We know by Theorem 2.4.(iii) that the sequence of M-stable mappings $(\phi_{T,n})_{n\geq 0}$ defined by $\phi_{T,n}(X) = p(n)^{-2}T^{*n}XT^n$ is weakly almost convergent, we will denote its limit by E_T. Now Proposition 1.2 shows that E_T is a completely positive projection. Notice also that E_T is an ultra-weakly continuous M-stable operator with $\widehat{E}_T = E_{T^*}$ and that $\|E_T\| \leq 1$. Let $\mathcal{S}$ be an abelian subset of M, which is dominated by the regular gauge p. We consider the abelian semigroup $\mathcal{E}(\mathcal{S})$ induced by the (abelian) set $\{E_T : T \in \mathcal{S}\}$.

Corollary 2.7. *Let M be a finite von Neumann algebra and let $\mathcal{S}$ be an abelian subset of M. Assume that $\mathcal{S}$ is dominated by a regular gauge p. Then the net $\mathcal{E}(\mathcal{S})$ weakly converges to a completely positive M-stable projection E satisfying the following properties*:

(i) *$E(T^*XT) = c_p^2 E(X)$ for any $T \in \mathcal{S}$ and $X \in M$;*

(ii) *$E(A^*XB) = A^*E(X)B$ for any pair $(A, B) \in (\mathcal{S}')^2$ and $X \in M$.*

Proof. We apply Proposition 2.6 to the net $\mathcal{E}(\mathcal{S})$. We deduce that $\mathcal{E}(\mathcal{S})$ converges weakly to $E \in \widehat{B}_\tau(M)$, for any $\tau \in \mathcal{T}(M)$. Since properties (i) and (ii) are true for the operators E_R ($R \in \mathcal{S}$) by Proposition 1.2, we see that the same properties hold for E. Q.E.D.

Proposition 2.8. *Let M be a finite von Neuman algebra, τ a faithful normal trace on M, and let $\mathcal{S}$ be an abelian subset of M which is dominated by a regular gauge p. Let us assume that the limit projection E of the net $\mathcal{E}(\mathcal{S})$ is such that $E(I)$ is injective. Then there exists an abelian set $\mathcal{S}_1$ of unitaries belonging to M such that for any $T \in \mathcal{S}$ there exists $U_T \in \mathcal{S}_1$ satisfying $\sqrt{E(I)}T = c_p U_T \sqrt{E(I)}$. Moreover, if F denotes the limit of the net $\mathcal{E}(\mathcal{S}_1)$, then we have the following properties*:

(i) $E(\sqrt{E(I)}X\sqrt{E(I)}) = \sqrt{E(I)}F(X)\sqrt{E(I)}$ *for any* $X \in M$;

(ii) $\sqrt{E(I)}\widehat{E}_\tau(X)\sqrt{E(I)} = \widehat{F}_\tau(\sqrt{E(I)}X\sqrt{E(I)})$ *for any* $X \in M$.

Proof. (i) First of all, observe that the equation $T^*E(I)T = c_p^2 E(I)$ and the injectivity of $E(I)$ imply that $\sqrt{E(I)}T$ is also injective. Taking the polar decomposition of $\sqrt{E(I)}T$, we see that there exists a unique isometry U_T such that $\sqrt{E(I)}T = c_p U_T \sqrt{E(I)}$. Since $U_T \in M$ and M is finite, it follows that U_T is unitary. The previous intertwining relations readily imply that the set $\mathcal{S}_1 := \{U_T : T \in \mathcal{S}\}$ is abelian.

Given $T \in \mathcal{S}$ and $X \in M$, we have

$$p(n)^{-2}T^{*n}\sqrt{E(I)}X\sqrt{E(I)}T^n = \frac{c_p^{2n}}{p(n)^2}\sqrt{E(I)}U_T^{*n}XU_T^n\sqrt{E(I)}$$

for every positive integer n. Taking a Banach limit we get the relation

$$E_T(\sqrt{E(I)}X\sqrt{E(I)}) = \sqrt{E(I)}E_{U_T}(X)\sqrt{E(I)}.$$

Now taking limits in the nets $\mathcal{E}(\mathcal{S})$ and $\mathcal{E}(\mathcal{S}_1)$ we get (i).

(ii) Since E and F are τ-M-stable, we can now get (ii) by the following computation. For any $(X, Y) \in M^2$ we have

$$\begin{aligned}\tau(X\sqrt{E(I)}\widehat{E}_\tau(Y)\sqrt{E(I)}) &= \tau(\sqrt{E(I)}X\sqrt{E(I)}\widehat{E}_\tau(Y)) = \tau(E(\sqrt{E(I)}X\sqrt{E(I)})Y) \\ &= \tau(\sqrt{E(I)}F(X)\sqrt{E(I)}Y) = \tau(X\widehat{F}_\tau(\sqrt{E(I)}Y\sqrt{E(I)})).\end{aligned}$$

Hence, we have $\sqrt{E(I)}\widehat{E}_\tau(Y)\sqrt{E(I)} = \widehat{F}_\tau(\sqrt{E(I)}Y\sqrt{E(I)})$ for any $Y \in M$. This completes the proof. Q.E.D.

III. Similarity

We say that an operator T is *asymptotically controlled* by a gauge p if T is compatible with p and satisfies the condition that $q'(\{\|T^n x\|^2/p(n)^2\}_n) > 0$, for every nonzero vector $x \in H$, where

$$q'(\xi) := \sup\left\{\liminf_k \frac{1}{m}\sum_{i=1}^{m}\xi(n_i + k) : m \in \mathbf{N}, n_1, \ldots, n_m \in \mathbf{N}\right\}$$

for any bounded real sequence ξ (see [Ker] for the role of this functional in the study of Banach limits).

For any real sequence $\xi \in \ell^\infty(\mathbf{N}^n)$ $(n \in \mathbf{N}, n > 1)$, let $Q_n\xi := \eta \in \ell^\infty(\mathbf{N}^{n-1})$, where $\eta(j_1, \ldots, j_{n-1}) := q'(\xi_{j_1,\ldots,j_{n-1}})$ with $\xi_{j_1,\ldots,j_{n-1}}(j) := \xi(j_1, \ldots, j_{n-1}, j)$. Let $\widetilde{Q}_n := Q_1 \circ \ldots \circ Q_{n-1} \circ Q_n$, where $Q_1 := q'$.

A set $\mathcal{F}$ of operators, acting on the Hilbert space H, is called asymptotically controlled by a gauge p, if every operator in $\mathcal{F}$ is compatible with p, and if for every nonzero vector $x \in H$ there exists $\rho(x) > 0$ such that

$$\widetilde{Q}_n\left(\left\{\frac{1}{p(j_1)^2 \cdots p(j_n)^2}\left\langle T_n^{*j_n} \cdots T_1^{*j_1} T_1^{j_1} \cdots T_n^{j_n} x, x\right\rangle\right\}_{j_1,\ldots,j_n=1}^{\infty}\right) \geq \rho(x)$$

is true for every $n \in \mathbf{N}$ and $T_1, \ldots, T_n \in \mathcal{F}$.

Remark 3.1. Let T be an operator compatible with a gauge p. Assume that T satisfies

$$\inf\{\|T^n x\| / p(n) : n \in \mathbf{N}\} > 0$$

for any nonzero x in H; then T is asymptotically controlled by p. In particular, power bounded operators of class $C_{1\cdot}$ (in the Sz.-Nagy–Foias terminology) are exactly operators which are asymptotically controlled by constant gauges.

Theorem 3.2. *Let $\mathcal{S}$ be an abelian set of operators which is contained in a finite von Neumann algebra. Assume that $\mathcal{S}$ is asymptotically controlled by a regular gauge p. Then, there exists an invertible operator A in M such that $r(T)^{-1}ATA^{-1}$ is a unitary operator for any $T \in \mathcal{S}$.*

Proof. Let τ be a faithful normal trace acting on M. Let E be the completely positive limit projection provided by Corollary 2.7. Since $\mathcal{S}$ is asymptotically controlled by the gauge p, we can infer by a short computation that, given any nonzero vector $x \in H$,

$$[l_{x,x}, E_{T_1} \circ E_{T_2} \circ \cdots \circ E_{T_n}(I)] \geq \rho(x)$$

is true for every choice of $T_1, \ldots, T_n \in \mathcal{S}, n \in \mathbf{N}$ with a $\rho(x) > 0$, whence

$$\langle E(I)x, x\rangle = [l_{x,x}, E(I)] \geq \rho(x) > 0.$$

Thus, the positive operator $E(I)$ is injective. Let us consider the associated set $\mathcal{S}_1$ and the corresponding limit operator F occurring in Proposition 2.8.

Set $X = E(I)$, $Y = \widehat{E}_\tau(I)$ and consider the positive operator $R = \sqrt{X}Y\sqrt{X} = \widehat{F}_\tau(X)$. Note that R commutes with U_T $(T \in \mathcal{S})$. Let P be a projection associated with the spectral decomposition of R (which still commutes with U_T). By the Cauchy–Schwarz Inequality, we get

$$\tau(PRP) = \tau(P\sqrt{X}Y\sqrt{X}P) \leq \sqrt{\tau(PXP)}\sqrt{\tau(P\sqrt{X}Y^2\sqrt{X}P)}. \tag{1}$$

Applying the properties of E and F described in Corollary 2.7 and Proposition 2.8, we infer that

$$\tau(PXP) = \tau(XP) = \tau(XF(P)) = \tau(\sqrt{X}F(P)\sqrt{X}) = \tau(E(\sqrt{X}P\sqrt{X}))$$

$$= \tau(\sqrt{X}P\sqrt{X}\widehat{E}_\tau(I)) = \tau(\sqrt{X}P\sqrt{X}Y) = \tau(PR) = \tau(PRP).$$

Now, note that the operator $C = \sqrt{X}P\sqrt{X}Y$ commutes with T^*, because we have

$$T^*C = T^*\sqrt{X}P\sqrt{X}Y = c_p\sqrt{X}U_T^*P\sqrt{X}Y = c_p\sqrt{X}PU_T^*\sqrt{X}Y$$

$$= (1/c_p)\sqrt{X}PU_T^*\sqrt{X}TYT^* = \sqrt{X}PU_T^*U_T\sqrt{X}YT^* = \sqrt{X}P\sqrt{X}YT^* = CT^*.$$

Using again Corollary 2.7 and Proposition 2.8 we obtain

$$\tau(P\sqrt{X}Y^2\sqrt{X}P) = \tau(Y\sqrt{X}P\sqrt{X}Y) = \tau(YC) = \tau(\widehat{E}_\tau(I)C) = \tau(\widehat{E}_\tau(C))$$

$$= \tau(\widehat{E}_\tau(C)E(I)) = \tau(\widehat{E}_\tau(I)CE(I)) = \tau(Y\sqrt{X}P\sqrt{X}YX)$$

$$= \tau(P\sqrt{X}YXY\sqrt{X}) = \tau(PR^2) = \tau(PR^2P).$$

Substituting these results into (1), we get

$$\tau(PRP) \leq \sqrt{\tau(PRP)}\sqrt{\tau(PR^2P)},$$

whence

$$\tau(PRP) \leq \tau(PR^2P). \tag{2}$$

Let K be a compact set contained in the interval $(0,1)$, and denote by P the spectral projection associated to K by the functional calculus of R. We thus have

$$PRP \geq (PRP)^2 = PR^2P \geq 0. \tag{3}$$

Combining (2) with (3) yields

$$\tau(PRP - PR^2P) = 0.$$

The operator $PRP - PR^2P$ is positive, so it is necessarily equal to 0. Therefore $Q = PRP$ is an orthogonal projection. But K is compact and contained in $(0,1)$, thus there exists $\rho \in (0,1)$ such that $Q \leq \rho I$. Consequently, we have $Q = 0$. It follows that

$$\sigma(R) \cap (0,1) = \emptyset.$$

The last step is devoted to show that $0 \notin \sigma(R)$. Let us denote by P the spectral projection associated to 0. We have $RP = 0$, thus

$$0 = \tau(RP) = \tau(\widehat{F}_\tau(X)P) = \tau(XF(P)) = \tau(XF(I)P) = \tau(XP) = \tau(PXP).$$

It follows that $PXP = 0$. Since X is injective, we deduce that $P = 0$. Finally, we see that R is invertible (actually, $\sigma(R) \subset [1,\infty)$), therefore X is also invertible. By Proposition 2.8 we know that $\sqrt{X}T = r(T)U_T\sqrt{X}$ ($c_p = r(T)$, see [Ker]). It follows that, for any $T \in \mathcal{S}$, $U_T = r(T)^{-1}ATA^{-1}$ is unitary, where $A = \sqrt{X}$ is invertible. Q.E.D.

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References

[Bea] B. Beauzamy, *Introduction to Operator Theory and Invariant Subspaces*, North-Holland, Amsterdam (1988).

[BCP] S. Brown, B. Chevreau and C. Pearcy, Contractions with rich spectrum have invariant subspaces, *J. Operator Theory*, **1** (1979), 123–136.

[CaFa] G. Cassier and T. Fack, Contractions in von Neumann algebras, *J. Funct. Anal.*, **135** (1996), 297–338.

[CaFa2] G. Cassier and T. Fack, On power-bounded operators in finite von Neumann algebras, *J. Funct. Anal.*, **141** (1996), 133–158.

[Dix] J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien (Algèbre de von Neumann)*, Gauthier-Villards, Paris (1969).

[Dix2] J. Dixmier, Les moyennes invariantes dans les semi-groupes et leurs applications, *Acta Sci. Math.* (*Szeged*), **12** (1950), 213–227.

[Lor] G. G. Lorentz, A contribution to the theory of divergent sequences, *Acta. Math.*, **80** (1948), 167–190.

[Ker] L. Kérchy, Operators with regular norm sequences, *Acta Sci. Math.* (*Szeged*), **63** (1997), 571–605.

[Ker2] L. Kérchy, Criteria of regularity for norm sequences, *Integral Equations Operator Theory*, **34** (1999), 458-477.

[Ker3] L. Kérchy, Representations with regular norm-behavior of discrete abelian semigroups, *Acta Sci. Math.* (*Szeged*), **65** (1999), 702–726.

[Ker4] L. Kérchy, Hyperinvariant subspaces of operators with non-vanishing orbits, *Proc. Amer. Math. Soc.*, **127** (1999), 1363–1370.

[Ker5] L. Kérchy, Unbounded representations of discrete abelian semigroups, *Progress in Nonlinear Differential Equations and Their Applications*, **42** (2000), 141–150.

[KeMü] L. Kérchy and V. Müller, Criteria of regularity for norm sequences. II, *Acta Sci. Math.* (*Szeged*), **65** (1999), 131–138.

[Sak] S. Sakai, *C^* algebras and W^* algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **60** (1971), Springer-Verlag, Berlin–New York.

[Nag] B. Sz.-Nagy, On uniformly bounded linear transformations in Hilbert space, *Acta Sci. Math.* (*Szeged*), **11** (1947), 152–157.

Institut Girard Desargues
UPRES-A 5028 Mathématiques
Université Claude Bernard Lyon I
69622 Villeurbanne Cedex, France
e-mail: cassier@desargues.univ-lyon1.fr